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STABILITY OF STEADY ROTATIONS OF A HEAVY GYROSTAT ABOUT ITS PRINCIPAL AXIS*

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Stability of steady rotations of a gyrostat about its principal axis is investigated with the use of the Arnol'd–Moser theorem /1,2/ extended to stationary motions /3, 4/. It is shown that steady rotations are stable for all parameter values that belong to the region where the necessary stability conditions are satisfied, except for some manifold of lesser dimension.

Sufficient conditions of stability of steady rotations of a gyrostat about its principal axis at an arbitrary angular velocity were obtained by Rumiantsev /5/. The case when the gyrostat can rotate about its principal axis only at some fixed velocity was analyzed in /6/. Sufficient conditions of stability of steady rotations of a gyrostat about any axis of the cone of steady rotation axes were obtained in /7,8/. The necessary conditions of stability were also indicated in /8/.

1. Steady rotations of the gyrostat about its principal axis.

The axes about which a gyrostat can rotate at constant velocity form in the space rigidly attached to it a cone mentioned by Kharlamov /9/. Analysis of that cone–locus of ends of angular velocity vectors of steady rotation-shows that the steady rotation of the gyrostat about its principal axis at arbitrary angular velocity $\boldsymbol{\omega}$ is only possible under condition that that axis carries the center of mass and the gyrostatic moment vector $\boldsymbol{\lambda}$ is directed along it. Let us investigate the stability of such motions relative to angular velocity projections $\omega_1, \omega_2, \omega_3$ and of the vector of the vertical ν_1, ν_2, ν_3 on the moving axes.

We define the gyrostat motion by Hamilton equations. On the assumption that the gyrostat center of mass lies on the first principal axis and that the gyrostic moment vector is directed along that same axis, the Hamiltonian for the gyrostat is of the form /10/

$$H=\frac{1}{2\sin^2\theta}\{a_1[(p_\psi-p_\varphi\cos\theta)\sin\varphi+p_\theta\cos\varphi\sin\theta-\lambda'\sin\theta]^2+ \tag{1.1}$$
$$a_2[(p_\psi-p_\varphi\cos\theta)\cos\varphi-p_\theta\sin\varphi\sin\theta]^2+\frac{a_3p_\varphi^2}{2}+\Gamma e\sin\varphi\sin\theta$$

where a_1, a_2, a_3 are components of the gyration tensor, Γ is the product of the gyrostat weight and the distance of the center of mass to the fixed point, λ' and e are projections on the first axis the gyrostatic moment vector and of the unit vector directed from the fixed point to the gyrostat center of mass, respectively.

To avoid the appearance of singularities in the equations of perturbed motions we determine in conformity with /3/ the investigated steady rotation of the gyrostat at angular velocity ω' by the following variables:

$$p_\theta=0,\quad p_\varphi=0,\quad p_\psi=\frac{\omega'}{a_1}+\lambda',\quad \theta=\frac{\pi}{2},\quad \varphi=\frac{\pi}{2},\qquad \psi=\omega't+\psi_0 \tag{1.2}$$

The steady rotation defined by equality (1.2) is the steady motion of a mechanical system with the Hamiltonian (1.1) and, since the presented system is in this case two-dimensional, we use for analyzing its stability the Arnol'd–Mozer theorem extended to steady motions /3/. Note that an investigation of steady rotation stability with respect to $\omega_1, \omega_2, \omega_3, \nu_1, \nu_2, \nu_3$ is equivalent to the investigation of stability of respective stationary motions with respect to $p_\theta, p_\varphi, p_\psi, \theta, \varphi$.

2. Expansion of the Hamiltonian in the neighborhood of steady rotation.

We pass to dimensionless variables x_1, x_2, y_1, y_2 and dimensionless time τ setting

$$(p_\theta, p_\varphi)=\sqrt{\Gamma/a_1}\,(x_1, x_2),\quad \left(\theta-\frac{\pi}{2},\ \varphi-\frac{\pi}{2}\right)=(y_1, y_2),\qquad \tau=t\sqrt{a_1\Gamma}$$

The equations of perturbed motion of the reduced system in the dimensionless form are of the form

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$$x_1^{\cdot} = -\frac{\partial H}{\partial y_1}, \quad x_2^{\cdot} = -\frac{\partial H}{\partial y_2}, \quad y_1^{\cdot} = \frac{\partial H}{\partial x_1}, \quad y_2^{\cdot} = \frac{\partial H}{\partial x_2} \tag{2.1}$$

$$\begin{aligned}
&H = H_2 + H_4 + \ldots \\
&2H_2 = ax_1^2 + bx_2^2 + (\omega^2 + \omega\lambda - e)\, y_1^2 + \{(\omega + \lambda) \times \\
&\quad [a(\omega + \lambda) - \omega] + - e\}\, y_2^2 + 2[a(\omega + \lambda) - \omega]\, x_1 y_2 + 2\omega x_2 y_1 \\
&24\, H_4 = (3\lambda^2 + 11\lambda\omega + 8\omega^2 + e)\, y_1^4 + [(\omega + \lambda)(4\omega + \\
&\quad 3\lambda - 4a(\omega + \lambda)) + e]\, y_2^4 + 6\,[(\omega + \lambda)(-2\omega - \lambda + \\
&\quad 2a(\omega + \lambda)) + e]\, y_1^2 y_2^2 + 12(1 - a)\, x_1^2 y_2^2 + \\
&\quad 12 x_2^2 y_1^2 + 4\,[4\omega + 3\lambda - 4a(\omega + \lambda)]\, x_1 y_2^3 + 4(5\omega + \\
&\quad 3\lambda)\, x_2 y_1^3 + 12(a - 1)(\omega + \lambda)\, x_1 y_1^2 y_2 + 12\,[-2\omega - \\
&\quad \lambda + 2a(\omega + \lambda)]\, x_2 y_1 y_2^2 + 24(a - 1)\, x_1 x_2 y_1 y_2 \\
&a = \frac{a_2}{a_1}, \quad b = \frac{a_3}{a_1}, \quad \omega = \frac{\omega'}{\sqrt{a_1 \Gamma}}, \quad \lambda = \sqrt{\frac{a_1}{\Gamma}}\,\lambda'
\end{aligned}$$

where the dot denotes differentiation with respect to τ. Parameters ω and λ can assume any arbitrary values, while parameters a and b must satisfy the conditions that follow from the inequality of the triangle for moments of inertia. The domain C of variation of parameters a and b is that of positive a and b bounded by curves $a = b(a + 1)$, $b = a(b + 1)$, $a = b(a - 1)$.

3. The necessary conditions of stability. To obtain the necessary conditions of stability we write the characteristic equation of the linearized system with function H_2

$$\mu^4 + \xi_1 \mu^2 + \xi_2 \xi_3 = 0 \tag{3.1}$$

$$\begin{aligned}
&\xi_1 = ab(\omega + \lambda)^2 - (a + b)(\omega + \lambda)\,\omega + 2\omega^2 - e(a + b) \\
&\xi_2 = \omega^2 (a - 1) + a\omega\lambda - ae, \quad \xi_3 = \omega^2 (b - 1) + b\omega\lambda - be
\end{aligned}$$

Consequently the necessary conditions of stability are of the form

$$\xi_1 > 0, \quad \xi_2 \xi_3 > 0 \tag{3.2}$$

$$\begin{aligned}
&\xi_1^2 - 4\xi_2\xi_3 = a^2 b^2 (\omega + \lambda)^4 - 2ab(a + b)(\omega + \lambda)^3 \omega + \\
&\quad (a + b)^2 (\omega + \lambda)^2 \omega^2 - 2abe(a + b)(\omega + \lambda)^2 + 2e(a + b)^2 \times \\
&\quad (\omega + \lambda)\,\omega + 8abe(\omega + \lambda)\,\omega - 8e(a + b)\,\omega^2 + (a - b)^2 > 0
\end{aligned}$$

In the space $0\xi_1\xi_2\xi_3$ region G, where the necessary stability conditions (3.2) are satisfied, consists of two subregions: G_1 and G_2 (Fig.1). The analysis of region G in the space of dimensionless parameters a, b, ω, λ is fairly complicated. It is, however, possible to state that, unlike in the case of the solid body /3/, conditions (3.2) are satisfied at all points of region C, although only for an appropriate selection of λ.

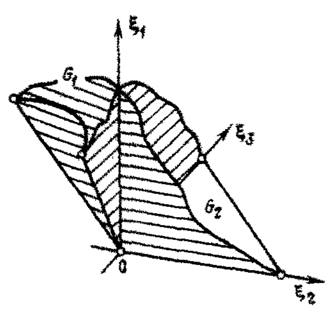

Fig.1

4. The sufficient conditions of stability. In region G_2 the quadratic form H_2 in expansion (2.1) is of fixed sign. This enables us to apply the Routh theorem with Liapunov's supplement and to state that the steady rotations that correspond to region G_2 are stable. In this region the sufficient conditions of stability $\xi_2 > 0$, $\xi_3 > 0$ are the same as obtained in /5/.

In region G_1 the quadratic form H_2 alternates. We denote the roots of the characteristic equation (3.1) by $\pm i\alpha_1$, $\pm i\alpha_2$ ($\alpha_1 > \alpha_2 > 0$) and write the canonical transformation which reduces Hamiltonian (2.1) to the form

$$H' = -i\alpha_1 p_1 q_1 + i\alpha_2 p_2 q_2 + \sum_{\nu_1 + \ldots + \nu_4 = 4} h_{\nu_1\nu_2\nu_3\nu_4} p_1^{\nu_1} q_1^{\nu_2} p_2^{\nu_3} q_2^{\nu_4} + \cdots \tag{4.1}$$

and obtain

$$\begin{aligned}
&x_1 = is_1 (p_1 - q_1) + ic_1 (q_2 - p_2), \quad y_1 = s_3 (p_1 + q_1) + c_3 (q_2 + p_2) \\
&x_2 = s_2 (p_1 + q_1) + c_2 (q_2 + p_2), \; y_2 = is_4 (p_1 - q_1) + ic_4 (q_2 - p_2) \\
&s_1 = \alpha_1 \{\alpha_1^2 - [a(\omega + \lambda) - \omega][b(\omega + \lambda) - \omega] + be\}\, w \\
&s_2 = \{(\omega^2 - \alpha_1^2)[a(w + \lambda) - \omega] - a\omega e\}\, w \\
&s_3 = \{a\alpha_1^2 - b\omega[a(\omega + \lambda) - \omega] + abe\}\, w \\
&s_4 = \alpha_1 [ab(\omega + \lambda) - \omega(a + b)]\, w
\end{aligned}$$

Formulas for c_1, c_2, c_3, c_4 are obtained from expressions for s_1, s_2, s_3, s_4 by the substitution of α_2 for α_1 and α_1 for α_2.

For resolving the question of stability of investigated motions it is necessary to calculate the determinant

$$D = -(\beta_{11}\alpha_2^2 + 2\beta_{12}\alpha_1\alpha_2 + \beta_{22}\alpha_1^2) \tag{4.2}$$

Coefficients $\beta_{11}, \beta_{22}, \beta_{12}$ are equal to the coefficients at $p_1^2q_1^2,\ p_2^2q_2^2,\ 2p_1q_1p_2q_2$ in the form $2iH'$, where the Hamiltonian H' is determined by formula (4.1). We have

$$\begin{aligned}
2\alpha_1\alpha_2(\alpha_1^2-\alpha_2^2)\beta_{11} = {}& {}^1/_2[3\lambda^2 + 11\lambda\omega + 8\omega^2 + e]s_3^4 + 2(5\omega + \\
& 3\lambda)s_2s_3^3 + {}^1/_2[(\omega+\lambda)(4\omega + 3\lambda - 4a(\omega+\lambda)) + e]s_4^4 + \\
& 2(4\omega + 3\lambda - 4a(\omega+\lambda))s_1s_1^3 + [(\omega+\lambda)(-2\omega - \lambda + \\
& 2a(\omega+\lambda)) + e]s_3^2s_4^2 + 6(1-a)s_1^2s_4^2 + 6s_2^2s_3^2 + \\
& 2(\omega+\lambda)(a-1)s_1s_4s_3^2 + 2(-2\omega - \lambda + 2a(\omega + \\
& \lambda))s_2s_3s_4^2 + 4(a-1)s_1s_2s_3s_4 \\
2\alpha_1\alpha_2(\alpha_1^2-\alpha_2^2)\beta_{12} = {}& {}^1/_2[3\lambda^2 + 11\lambda\omega + 8\omega^2 + e]s_3^2c_3^2 + \\
& (s_3c_2 + c_3s_2)^2 + {}^1/_2[(\omega+\lambda)(-2\omega - \lambda + 2a(\omega+\lambda)) + \\
& e](s_3^2c_4^2 + c_3^2s_4^2) + 2s_2s_3c_2c_3 + {}^1/_2[(\omega+\lambda)(4\omega + 3\lambda - \\
& 4a(\omega+\lambda)) + e]s_4^2c_4^2 + (1-a)((s_4c_1 + s_1c_4)^2 + \\
& 2s_1s_4c_1c_4) + (4\omega + 3\lambda - 4a(\omega+\lambda))(s_1c_4 + s_4c_1)s_4c_4 + \\
& (5\omega + 3\lambda)(s_2c_3 + s_3c_2)s_3c_3 + (a-1)(\omega+\lambda)(s_1s_4c_3^2 + \\
& c_1c_4s_3^2) + (-2\omega - \lambda + 2a(\omega+\lambda))(s_2s_3c_4^2 + c_2c_3s_4^2) + \\
& 2(a-1)(s_2s_3c_1c_4 + s_1s_4c_2c_3)
\end{aligned}$$

The expression for β_{22} is obtained from the formula for β_{11} by interchanging the positions of s_k and c_k.

The determinant (4.2) was calculated in /3/ for $\lambda = 0,\ a = 1$ and shows that $D(a, b, \lambda, \omega) \not\equiv 0$, hence the equality $D(a, b, \lambda, \omega) = 0$ isolates in the space $0ab\lambda\omega$ some manifold. The resonance relations $\alpha_1 = 0,\ \alpha_2 = 0,\ \alpha_1 = \alpha_2,\ \alpha_1 = 3\alpha_2$ also separate some manifolds in the space $0ab\lambda\omega$. The steady rotations that correspond to the separated manifolds are not considered here. As regards the remaining steady solutions that belong to region G_1, we conclude on the basis of the Arnol'd–Mozer theorem extended to stationary motions /3/ that such motions are Liapunov stable.

Hence the following theorem has been proved.

Theorem. Let a gyrostat steadily rotate about its principal axis which passes through the center of mass and along which is directed the gyrostatic moment vector. Then in the extended parametric space $0ab\lambda\omega$ the stability region is region G where the necessary conditions of stability are satisfied and from which are excluded the manifolds that correspond to resonance relations and to the condition for determinant to be zero.

The comparison of the obtained here results with those of investigation of solid body steady rotations /3/ makes it possible to assert that the presence in a body of a rotor rotating at suitable velocity has a stabilizing effect on the motions of the body. The unsteady rotation of a body about its middle principal axis can be stabilized by an appropriate selection of the gyrostatic moment. Moreover, any unsteady rotation of a soid body can be made steady by a suitable selection of the gyrostatic moment. This follows from that at fairly large absolute values of the kinetic moment vector the sufficient stability conditions are satisfied for any fixed vector of angular velocity and any moment of inertia.

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